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RAYLEIGH'S METHOD APPLIED TO A CONDUCTING LIQUID DROP  
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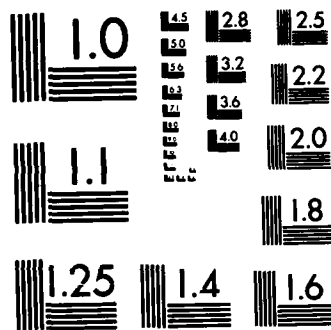


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Rayleigh's Method Applied to a Conducting Liquid Drop in the Presence of a Point Charge

by Clyde Morrison  
Richard Leavitt

ABSTRACT

The method that Rayleigh formulated in 1879 is used to determine the effect of a point charge on the natural frequencies of a charged drop. Explicit expressions are derived for the resonant frequencies and the deformation of the charged drop. The results are expressed in terms of sums over particular types of Clebsch-Gordon coefficients and more complicated sums. Further work is necessary before practical application of the results can be made.

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## INTRODUCTION

The purpose of this work is to extend our previous method of solution to problems involving drops in an electric field<sup>(1)</sup> to charged droplets. Consequently, we summarize some of our more recent results on the study of the dynamics of liquid droplets.

The methods employed here are consistent with the method employed by Lord Rayleigh<sup>(2)</sup> in his original treatment of the dynamics of liquid droplets. We begin with a brief rederivation of Rayleigh's results for uncharged and charged conducting droplets.<sup>(3)</sup> This is followed by a more detailed derivation of the problem of a charged droplet in the presence of a point charge. Extensive attention is given to this latter result, since it appears to be new. Many of the results given here may have application to the formation of clouds,<sup>(4-8)</sup> the physical aspects of drop formation,<sup>(9)</sup> the stability of electrified surfaces,<sup>(10)</sup> and the electrical dispersion of liquid aerosols.<sup>(11)</sup>

## RAYLEIGH'S RESULT

In his original paper, Rayleigh<sup>(2)</sup> assumed that the distance,  $r$ , from the center to a point on the surface of the droplet can be expanded in a Legendre series as

$$r(\theta, t) = a_0(t) + \sum_k a_k(t) P_k(\cos \theta), \quad (1)$$

where we have assumed that the drop is symmetric around the  $z$  axis of the drop. At this point the  $z$  axis can be chosen in any direction, but later, when we include the electromagnetic energy, the electric field will be assumed



along this axis. The prime on the sum will be used throughout to denote the absence of the  $k = 0$  term. The volume of the drop is given by

$$V = \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^{r(\theta)} r'^2 \sin \theta dr' \quad (2)$$

Since the drop is symmetric around  $z$  and with  $\mu = \cos \theta$ , we have

$$V = \frac{2\pi}{3} \int_{-1}^1 r^3 d\mu \quad (3)$$

From (1) we have

$$r^3 = a_0^3 + 3a_0^2 \sum' a_k P_k + 3a_0 \sum_{kk'}' a_k a_{k'} P_k P_{k'} \quad (4)$$

and we use this result in (3) to obtain

$$V = \frac{2\pi}{3} \left[ 2a_0^3 + 3 \cdot 2 a_0 \sum_k' \frac{a_k^2}{2k+1} \right] \quad (5)$$

where we have used\*

$$\int_{-1}^1 P_\ell P_{\ell'} d\mu = \frac{2}{2\ell+1} \delta_{\ell\ell'} \quad (6)$$

and we have terminated the approximation at the square of  $a_k$ . We will follow this procedure throughout; that is, we assume  $|a_k| \ll a_0$  and products of more than two  $a_k$  are ignorable ( $k \neq 0$ ). We assume the fluid is incompressible with equilibrium radius "a" so that the volume (5) is a constant,  $4\pi a^3/3$ ; that is,

$$a^3 = a_0^3 + 3 a_0 \sum_k' \frac{a_k^2}{2k+1} \quad (7)$$

\*See the appendix for a number of useful relations involving Legendre polynomials.



or

$$a = a_0 \left[ 1 + \frac{1}{a_0^2} \sum_k \frac{a_k^2}{2k+1} \right] , \quad (8)$$

then

$$a_0 = a - 1/a \sum_k \frac{a_k^2}{2k+1} . \quad (9)$$

The result given by (9) serves as a constraint on the variables  $a_k$ .

The potential energy,  $U_s$ , of the drop due to the surface tension,  $\gamma$ , is the surface area of the drop multiplied by the surface tension, or

$$U_s = \gamma \int_0^{2\pi} d\phi \int r \sin \theta ds , \quad (10)$$

where  $ds$  is the arc length along the surface given by

$$ds^2 = dr^2 + r^2 d\theta^2 .$$

Then

$$ds = d\theta \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]^{1/2} . \quad (11)$$

Using the result (11) in (10), we have

$$U_s = 2\pi\gamma \int_{-1}^1 r \left[ r^2 + (1 - \mu^2) \left( \frac{dr}{d\mu} \right)^2 \right]^{1/2} d\mu , \quad (12)$$

where we have used the relation

$$\frac{d}{d\theta} = -\sin \theta \frac{d}{d\mu} .$$

Expanding the integrand in (12) we have

$$U_s = 2\pi\gamma \int_{-1}^1 \left[ r^2 + \frac{(1-\mu^2)}{2} \left( \frac{dr}{d\mu} \right)^2 \right] d\mu , \quad (13)$$



and since  $\frac{dr}{du}$  does not involve  $a_0$ , we need consider no higher terms. The integrals in (13) can be evaluated simply to give (see the Appendix)

$$U_s = 2\pi\gamma \left[ 2a_0^2 + \sum_k' \frac{[k(k+1)+2]}{2k+1} a_k^2 \right] , \quad (14)$$

and if we use the constraint given in (9), we have

$$U_s = 2\pi\gamma \left[ 2a^2 + \sum_k' \frac{(k-1)(k+2)}{2k+1} a_k^2 \right] , \quad (15)$$

which gives the potential energy due to surface tension correct through terms of order  $a_k^2$ .

To calculate the kinetic energy,  $T$ , we need to evaluate the integral

$$T = \int \frac{1}{2} \rho v^2 d\tau , \quad (16)$$

where  $\rho$  is the density and  $v$  is the velocity. Since we are assuming that the fluid is incompressible and that there are no sources or sinks within the drop, we have

$$\nabla \cdot \vec{v} = 0 ; \quad (17)$$

and further, if we assume

$$\nabla \times \vec{v} = 0 , \quad (18)$$

then  $\vec{v}$  can be derived from a potential function,  $\psi_v$ , such that

$$\vec{v} = \nabla \psi_v , \quad (19)$$

and from (17) we have

$$\nabla^2 \psi_v = 0 . \quad (20)$$

Using (19) in (16) we have

$$T = \frac{1}{2} \rho \int (\nabla \psi_v)^2 d\tau , \quad (21)$$



and by converting the integral over the volume to a surface integral, we have

$$T = \frac{1}{2} \rho \int \psi_v (\nabla \psi_v) \cdot d\vec{\sigma} . \quad (22)$$

Assuming that the area element  $d\vec{\sigma}$  is approximately along  $\hat{r}$  (corrections are of higher order), we can write  $d\vec{\sigma} = \hat{r} a^2 d\phi d\mu$ , where  $\phi$  = the azimuthal angle and  $\hat{r}$  = the unit vector along the radius; then

$$T = \pi \rho a^2 \int_{-1}^1 \psi_v \frac{d\psi_v}{dr} d\mu . \quad (24)$$

The solution of (19) appropriate for our problem is

$$\psi_v = \sum_n \beta_n r^n P_n(\mu) , \quad (25)$$

where we have assumed that the potential is evaluated at the surface. Using (25) in (24) we have

$$T = 2\pi \rho a^2 \sum \frac{n}{2n+1} a^{2n-1} \beta_n^2 . \quad (26)$$

The  $\beta_n$  in (26) can be evaluated by equating the velocity at the surface, given by  $\frac{d\psi_v}{dr}$ , to  $\dot{r}$  from (1), or

$$\dot{a}_n = n a^{n-1} \beta_n . \quad (27)$$

Then (26) becomes

$$T = 2\pi \rho a^3 \sum \frac{\dot{a}_n^2}{n(2n+1)} . \quad (28)$$

The results given in (28) and in (15) are sufficient to form the Lagrangian ( $L = T - U$ ) in the absence of other energy sources. Rayleigh<sup>(2)</sup> uses these



results to obtain the equation of motion for the  $a_n(t)$  by treating the  $a_n$  as generalized coordinates. We will put off doing this until the electromagnetic energy is contained in the Lagrangian.

The particular problem we wish to solve here is for a charged conducting sphere of equilibrium radius  $a$  with total charge  $Q$ . This problem was also solved by Rayleigh,<sup>(12)</sup> but by a slightly different technique than was used in his original paper. Since we are interested here in extending his original technique to a new problem, we shall include more detail than in our previous discussion.

From Maxwell's equations we have

$$\nabla \times \vec{E} = 0, \quad (29)$$

or

$$\vec{E} = -\nabla \psi.$$

Then from

$$\nabla \cdot \vec{E} = 0, \quad (30)$$

we have

$$\nabla^2 \psi = 0. \quad (31)$$

Thus, the potential  $\psi$  is a solution of Laplace's equation. The appropriate solution of (31) for our problem for  $r > r(\theta, t)$  is

$$\psi = \sum_n \frac{A_n}{r^{n+1}} P_n. \quad (32)$$

The total charge on the sphere,  $Q$ , is assumed constant; thus,  $A_0 = Q$ . This result ( $A_0 = Q$ ) is simply proven by assuming a very large sphere surrounding the charged drop and employing Gauss' theorem at the surface of that sphere. The other  $A_n$  must be found from the boundary conditions of the problem. Once the  $A_n$  have been determined, the electromagnetic energy is given by<sup>(13)</sup>



$$U_E = \frac{1}{2} \int \rho \psi d\sigma, \quad (33)$$

where  $\rho$  is the surface charge density,  $\psi$  is the potential at the surface, and the integral covers the surface of the sphere. Since the potential on a conducting surface is constant,  $V$ , say, then (33) becomes

$$U_E = \frac{1}{2} QV, \quad (34)$$

where  $Q = \int \rho d\sigma$  and is a constant by assumption. The potential at the surface,  $V$ , is however a function of the variation of the shape of the surface, as will become apparent.

To proceed further it is convenient to introduce the notation

$$r(\theta, t) = a_0 + \delta \sum a_k P_k,$$

$$v = \sum \delta^\nu v^{(\nu)},$$

and

$$A_n = \sum \delta^\nu A_n^{(\nu)}, \quad (35)$$

which serves to keep track of the order of the corrections. In the final result we let  $\delta = 1$ , since

$\delta$  only serves as an artifice to keep the terms in order. The single boundary condition needed is

$$v = \frac{Q}{r} + \sum_n' \frac{A_n}{r^{n+1}} P_n, \quad (36)$$

where  $r$  is the function of  $\theta$  given in (35). The coefficients  $A_n^{(\nu)}$  in (35) can be determined by using (35) in (36) and equating powers of  $\delta$  on each side. If this is done, the results through  $\delta^2$  are

$$v^{(0)} = \frac{Q}{a_0},$$



$$A_n^{(0)} = 0 ,$$

$$A_n^{(1)} = Q a_0^{n-1} a_n ,$$

$$V^{(1)} = 0 ,$$

and

$$V^{(2)} = \sum' - \frac{(n+1)}{a_0^{n+2}} \frac{a_n A_n^{(1)}}{(2n+1)} + \frac{Q}{a_0^3} \sum' \frac{a_n^2}{2n+1}$$

or

$$V^{(2)} = - \frac{Q}{a_0^3} \sum' \frac{n a_n^2}{(2n+1)} . \quad (37)$$

To obtain this result, the equations given in the appendix were used. The electromagnetic energy through second order is given by using (37) in (34) or

$$U_E = 1/2 Q [V^{(0)} + V^{(2)}] . \quad (38)$$

The result for  $V^{(0)}$  can be reduced further by using the constraint, (9), in (37) to give

$$V^{(0)} = \frac{Q}{a} \left[ 1 + \frac{1}{a^2} \sum' \frac{a_n^2}{2n+1} \right] . \quad (39)$$

The result (39) along with (15) and (28) is sufficient to form the Lagrangian for the system, that is

$$L = T - U_S - U_E ,$$

resulting in

$$L = 2\pi p a^3 \sum' \frac{\dot{a}_n^2}{n(2n+1)} - 2\pi\gamma \sum' \frac{(n-1)(n+2)}{(2n+1)} a_n^2 + \frac{Q^2}{2a^3} \sum' \frac{(n-1)}{2n+1} a_n^2 , \quad (40)$$



where the constant terms in (15) and in (39) have been ignored.

The equation of motion for  $a_n$  is Lagrange's equation,  $\frac{d}{dt} \frac{\partial L}{\partial \dot{a}_n} - \frac{\partial L}{\partial a_n} = 0$ , which becomes

$$\ddot{a}_n + \omega_n^2 a_n = 0, \quad (41)$$

where

$$\omega_n^2 = \frac{n(n-1)}{\rho a^3} \left[ \gamma(n+2) - \frac{Q^2}{4\pi a^3} \right],$$

which is the result derived by Rayleigh<sup>(12)</sup> expressed in the same form. As was pointed out by Rayleigh, the system becomes unstable for  $\omega_n^2 < 0$ , which occurs for values of  $Q$  such that

$$\frac{Q^2}{4\pi a^3} > \gamma(n+2), \quad (42)$$

which for the lowest mode ( $n = 2$ ) requires that  $\frac{Q^2}{4\pi a^3} > 4\gamma$ . The charge on the drop acts so as to reduce the surface tension so that only the higher modes are stable. The result given in (42) has been used by numerous workers as a starting point for the discussion of not only charged drops but also uncharged drops in an external uniform electric field, one of the more recent being the article by Smith.<sup>(13)</sup>

If in (42) we replace the charge by the electric field at the surface of the equilibrium sphere ( $Q = a^2 E$ ), then we can write

$$\frac{E^2 a}{\gamma} > n + 2, \quad (43)$$

and for the fundamental mode ( $n = 2$ ) we have

$$E \frac{a}{\gamma} = 2,$$



for the onset of instability. Frequently, in reporting experimental results of drops in electric fields, the field  $E$  in (43) is taken as the external field and the constant  $C_0$ , determined in the equation

$$E \frac{\bar{a}}{\gamma} = C_0, \quad (44)$$

by experimental means. Such a procedure was adapted by Wilson and Taylor in their experiments with soap bubbles, where they evaluated  $C_0 = 1.61 \pm$  as the critical value in (44).

#### CONDUCTING DROP IN THE FIELD OF A POINT CHARGE

We assume as before that the conducting drop of charge,  $Q$ , equilibrium radius  $a$  is at the origin of the coordinate system. The point charge,  $Q_1$ , is located on the positive  $z$  axis at a distance  $R$  from the origin. The energy of the system corresponding to (33) is

$$U_E = \frac{1}{2} QV + \frac{1}{2} Q_1 \psi(R), \quad (45)$$

where, as before,  $V$  is the potential at the surface of the sphere and  $\psi(R)$  is the electric potential at the point charge. The expressions for the kinetic energy, (28), and the surface energy, (15), remain the same as before.

The potential  $\psi$  appropriate for the problem is

$$\psi = \sum \frac{A_n}{r^{n+1}} P_n + Q_1 \sum \frac{r^n}{R^{n+1}} P_n, \quad (46)$$



where in the second term we have assumed  $r < R$ , which is the region of the interest. It is convenient to rewrite (46) as

$$\psi = \frac{Q}{r} + \frac{Q_1}{R} + \sum_n \frac{A_n}{r^{n+1}} P_n + Q_1 \sum_n \frac{r^n}{R^{n+1}} P_n, \quad (47)$$

so that all the sums have the terms  $n = 0$  omitted. To evaluate the constants in (47), it is convenient to assume the same "bookkeeping procedure" as given in (35) to expand each side of the equation

$$V = \psi(r)|_{r=r(\theta,t)} = \frac{Q}{r} + \frac{Q_1}{R} + \sum_n \frac{A_n}{r^{n+1}} P_n + Q_1 \sum_n \frac{r^n}{R^{n+1}} P_n, \quad (48)$$

and equate the coefficients of each order. The terms in the potential are then given by

$$V^{(0)} = \frac{Q_1}{R} + \frac{Q}{a_0}, \quad (49)$$

$$V^{(1)} = Q_1 \sum_n \frac{n a_0^{n-1} a_n}{(2n+1) R^{n+1}} - \sum_n \frac{(n+1) a_n A_n^{(0)}}{(2n+1) a_0^{n+2}}, \quad (50)$$

$$V^{(2)} = \frac{Q}{a_0} \sum_n \frac{a_n^2}{2n+1} - \sum_n \frac{(n+1) a_n A_n^{(1)}}{(2n+1) a_0^{n+2}} + \sum_n \frac{(n+1)(n+2) A_n^{(0)}}{2(2n+1) a_0^{n+3}} \sum_{k,k'} a_k a_{k'} \langle kk'|n \rangle^2$$

$$+ Q_1 \sum_n \frac{n(n-1)}{2(2n+1)} \frac{a_0^{n-2}}{R^{n+1}} \sum_{kk'} a_k a_{k'} \langle kk'|n \rangle^2, \quad (51)$$

where the Clebsch-Gordan coefficients (C-G coefficients)  $\langle kk'|n \rangle^2$  are given in the appendix. The corresponding expression for the  $A_n$  are given by

$$A_n^{(0)} = - \frac{Q_1}{R^{n+1}} a_0^{2n+1}, \quad (52)$$

$$A_n^{(1)} = a_0^{n-1} Q a_n + a_0^{n-1} \sum_{lk} \frac{(l+1) A_l^{(0)} a_k}{a_0^l} \langle kl|n \rangle^2 - a_0^n Q_1 \sum_{lk} \frac{l a_0^l}{R^{l+1}} a_k \langle kl|n \rangle^2,$$



(53)

$$A_n^{(2)} = -a_0^{n-2} Q \sum_{kk'} a_k a_{k'} \langle kk'|n \rangle^2 + a_0^n \sum_{\ell k} \frac{(\ell+1)A_\ell^{(1)} a_k}{a_0^{\ell+1}} \langle k\ell|n \rangle^2 \\ - a_0^{n-1} \sum_{\ell k k'} \left[ \frac{(\ell+1)(\ell+2)A_\ell^{(0)}}{2a_0^{\ell+1}} + \frac{\ell(\ell-1)a_0^\ell Q_1}{2R^{\ell+1}} \right] a_k a_{k'} \sum_v \langle kk'|v \rangle^2 \langle v\ell|n \rangle^2 .$$

By combining (49) through (54), we obtain

$$v^{(0)} = \frac{Q_1}{R} + \frac{Q}{a_0} , \quad (55)$$

$$v^{(1)} = \frac{Q_1}{2} \sum_{a_0} \left( \frac{a_0}{R} \right)^{n+1} a_n , \quad (56)$$

$$v^{(2)} = -\frac{Q}{a_0^3} \sum_n \frac{n a_n^2}{2n+1} + \frac{Q_1}{a_0^3} \sum_{nk\ell} \ell a_k a_\ell \left( \frac{a_0}{R} \right)^{n+1} \langle k\ell|n \rangle^2 , \quad (57)$$

$$A_n^{(1)} = a_0^{n-1} Q a_n - a_0^{n-1} Q_1 \sum_{\ell k} \frac{(2\ell+1)a_0^{\ell+1} a_k}{R^{\ell+1}} \langle k\ell|n \rangle^2 , \quad (58)$$

$$A_n^{(2)} = a_0^{n-2} Q \sum_{\ell k} \ell a_\ell a_k \langle k\ell|n \rangle^2 - Q_1 a_0^{n-2} \sum_{\ell k k'} (2\ell+1) \left( \frac{a_0}{R} \right)^{\ell+1} a_k a_{k'} T_{k'k\ell} ,$$

and

$$T_{k'k\ell} = \sum_v v \langle k'\ell|v \rangle^2 \langle vk|n \rangle^2 . \quad (59)$$

These results are all that are necessary to obtain the electrostatic energy through terms of order  $a_k^2$ . In (56) through (58),  $a_0$  can be replaced by "a", but in (55) the constraining equation, (9), must be used to cast this term in final form. If this is done we have  $v = v^{(0)} + v^{(1)} + v^{(2)}$  and if  $W_1 = 1/2 QV$ , the first part of the energy given in (45) is



$$W_1 = \frac{Q}{2} \left[ \frac{Q_1}{R} + \frac{Q}{a} + \frac{Q_1}{a^2} \sum_n x^{n+1} a_n - \frac{Q}{a^3} \sum_n \frac{(n-1)}{2n+1} a_n^2 + \frac{Q_1}{a^3} \sum_{kl} a_{kl} a_k G_{kl}(x) \right],$$

where

$$G_{kl}(x) = \sum_{n=1}^{\infty} x^{n+1} \langle kl|n \rangle^2,$$

and

$$x = a/R.$$

This result, (60), is in final useable form for Lagrange's equations, but the second part of the electrostatic energy given by (45) needs more reduction before it can be used. From the second term in (45) we have

$$W_2 = \frac{1}{2} Q_1 \psi(R), \quad (61)$$

where the terms in  $\psi(r)$  that become infinite at  $r = R$  are removed. From (46) and (48) we have

$$W_2 = \frac{1}{2} Q_1 \left[ \frac{Q}{R} + \sum_n' \frac{A_n}{R^{n+1}} \right], \quad (62)$$

where in the second term we have used the relation  $P_n(\cos\theta) = 1$  for  $\theta = 0$ , since we are assuming that the charge  $Q_1$  lies on the positive  $z$  axis at a distance  $R$  from the origin. As in our previous procedure, we assume

$$W_2 = W_2^{(0)} + W_2^{(1)} + W_2^{(2)}, \quad (63)$$

so that from (62) we have

$$W_2^{(0)} = \frac{1}{2} Q_1 \left[ \frac{Q}{R} + \sum_n' \frac{A_n^{(0)}}{R^{n+1}} \right], \quad (64)$$



$$w_2^{(1)} = \frac{1}{2} Q_1 \sum_n \frac{A_n^{(1)}}{R^{n+1}}, \quad (65)$$

and

$$w_2^{(2)} = \frac{1}{2} Q_1 \sum_n \frac{A_n^{(2)}}{R^{n+1}}. \quad (66)$$

Each of the terms, (64), (65), and (66), require considerable effort to reduce them to useable results.

The first, (64), is the simplest; when (52) is used, we have

$$w_2^{(0)} = \frac{1}{2} \frac{Q_1}{R} - \frac{Q_1^2}{2a} \left[ \frac{x^4}{1-x^2} - \frac{x^4(3-x^2)}{(1-x^2)^2} \frac{1}{a^2} \sum_n \frac{a_n^2}{2n+1} \right], \quad (67)$$

where  $x = a/R$ , and we have used the constraint (9). If in (65) we use (58), we have

$$w_2^{(1)} = \frac{Q_1}{2a^2} \sum_n x^{n+1} a_n - \frac{Q_1^2}{2a^2} \sum_{n,k} (2k+1) x^{k+1} a_n G_{kn}(x), \quad (68)$$

where  $G_{kn}(x)$  is as defined in (60). The last term, (66), can be obtained by substituting (59) into (66), which gives

$$w_2^{(2)} = \frac{Q_1}{2a^3} \sum_{\ell k} \ell a_\ell a_k G_{\ell k}(x) - \frac{Q_1^2}{2a^3} \sum_{k\ell m} m(2m+1) a_k a_\ell G_{\ell m}(x) G_{mk}(x). \quad (69)$$

With the result given in (69), we have completed all the terms in the potential energy that are necessary to form the Lagrangian of the problem. Unlike most of the previous results, there appears to be no advantage in combining (67), (68) and (69), since corresponding terms do not exist. The Lagrangian can be written

$$L = T - U, \quad (70)$$



with

$$U = U_S + U_E \quad , \quad (71)$$

where  $T$  is given in (28),  $U_S$  is given in (15) and

$$U_E = W_1 + W_2^{(0)} + W_2^{(1)} + W_2^{(2)} \quad . \quad (72)$$

The equation of motion for the generalized coordinate  $a_p$  is given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{a}_p} - \frac{\partial L}{\partial a_p} = 0 \quad , \quad (73)$$

which can be written as

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{a}_p} + \frac{\partial U_S}{\partial a_p} = - \frac{\partial U_E}{\partial a_p} \quad . \quad (74)$$

With the appropriate substitutions we have

$$\begin{aligned} & 4\pi\rho a^3 \frac{\ddot{a}_p}{p(2p+1)} + 4\pi\gamma \frac{(p-1)(p+2)}{2p+1} a_p \\ &= \frac{Q^2}{a^3} \frac{(p-1)}{2p+1} a_p - \frac{Q_1}{a^3} (2p) G_{pp}(x) a_p \\ & \quad - \frac{Q_1^2}{a^3} \frac{x^4(3-x^2)}{(1-x^2)^2} \frac{a_p}{2p+1} \\ & \quad + \frac{Q_1}{a^3} a_p \sum_m (2m+1) G_{pm}^2(x) \\ & \quad + F_0 + F_1 \quad , \end{aligned} \quad (75)$$

where



$$F_0 = \frac{Q_0 Q_1}{a^2} x^{p+1} + \frac{Q_1^2}{2a^2} \sum_k (2k+1) x^{k+1} G_{kp}(x) ,$$

and

$$F_1 = -\frac{Q_0 Q_1}{a^3} \sum_l (l+p) a_l G_{lp}(x) + \frac{Q_1^2}{a^3} \sum_l \sum_m m(2m+1) a_l G_{lm}(x) G_{mp}(x) .$$

The terms on the right side of (75) have been grouped in a particular manner. The first four terms multiply  $a_p$  and act as a restoring (or repulsive, depending on the sign of  $Q_1$ ) force. The term  $F_0$  acts as a constant force causing a displacement from the equilibrium point (normally, the equilibrium point is at  $a_p = 0$ ), and the term  $F_1$  acts to couple  $a_p$  to the other modes of vibration.

If we ignore  $F_0$  and  $F_1$  on the right side of (75), we can write

$$\ddot{a}_p + \omega_p^2 a_p = 0 , \quad (76)$$

where

$$\rho a^3 \omega_p^2 = \gamma p(p-1)(p+2) - U_1 p(p-1) + U_2 \frac{x^4(3-x^2)}{(1-x^2)^2} p + U_{12} (2p^2)(2p+1) G_{pp}(x) - U_2 p(2p+1) \sum_m m(2m+1) G_{pm}^2(x) , \quad (77)$$

where  $U_1 = \frac{Q^2}{4\pi a^3}$ ,  $U_2 = \frac{Q_1^2}{4\pi a^3}$ , and  $U_{12} = \frac{Q_0 Q_1}{4\pi a^3}$ . The first two terms on the right side of (77) are just  $\rho a^3$  times the frequency of a charged drop as derived originally by Rayleigh,<sup>(12)</sup> and are given in (41). Since  $F_0$ ,  $F_1$ ,  $U_2$ , and  $U_{12}$  vanish when  $Q_1 = 0$  the result (77) becomes identical to (41), as it should. If, on the other hand  $Q = 0$ , we have, from (77),



$$\rho a^3 \omega_p^2(Q=0) = \gamma p(p-1)(p+2) + U_2 \frac{x^4(3-x^2)}{(1-x^2)^2} p - U_2 p(2p+1) \sum_m m(2m+1) G_{pm}^2(x), \quad (78)$$

which can cause either an increase or decrease in the resonant frequency of a free drop [ $\rho a^3 \omega_p^2 = \gamma p(p-1)(p+2)$ ] through the rather complicated dependence of the last two terms on  $x$ . The frequency shift given in (78) does not depend on the sign of the point charge  $Q_1$  as it should not; however, when the charge  $Q$  is not zero, then (77) shows that the sign of the charge  $Q_1$  relative to  $Q$  is significant, as apparent by the terms involving  $U_{12}$ .

Many of the less obvious results of (77) will have to await considerable computational investigation before quantitative statements can be made. However, it is possible that further analytical work can be done on the  $G_{pp}(x)$  and the sums involving these functions. They seem to be expressible in terms of terminating hypergeometric functions. This can be seen by expressing the CG-coefficients in  $G_{pp}(x)$  in Racah's<sup>(15,16)</sup> closed form and using expressions for the hypergeometric series and their identities given in Rainville.<sup>(17)</sup>



# APPENDIX

During the course of the development of the theory a number involving relations of Legendre polynomials are needed. We abbreviate  $P_\ell(\mu) = P_\ell$  and  $\frac{d}{d\mu} P_\ell(\mu) = P'_\ell$  in the results.

$$\int_{-1}^1 P_\ell P_k d\mu = \frac{2}{2\ell+1} \delta_{\ell k} . \quad (A1)$$

$$\int_{-1}^1 (1 - \mu^2) P'_\ell P'_k d\mu = \frac{2\ell(\ell+1)}{2\ell+1} \delta_{\ell k} . \quad (A2)$$

$$\int_{-1}^1 (1 - \mu^2) P_n P'_k P'_\ell d\mu = \frac{[\ell(\ell+1)+k(k+1)-n(n+1)]}{2n+1} \langle \ell k | n \rangle^2 . \quad (A3)$$

$$[\langle \ell k | n \rangle = \langle \ell(0)k(0) | n(0) \rangle = C(\ell k n; 00)] ,$$

where  $\langle \ell k | n \rangle$  is a Clebsch-Gordan coefficient. Of these three results, the last (A3) is the only difficult one to derive. A thorough discussion of Clebsch-Gordan coefficients is given by Rose<sup>(16)</sup> or Brink and Satchler,<sup>(18)</sup> but is more general than is necessary here. In general, the simple relations given here can be derived from

$$P_k P_\ell = \sum \langle \ell k | n \rangle^2 P_n . \quad (A4)$$

Thus,

$$\langle \ell k | n \rangle^2 = \frac{(2n+1)}{2} \int_{-1}^1 P_n P_k P_\ell d\mu , \quad (A5)$$

which is a special case of Gaunt's<sup>(19)</sup> formula.

Other properties which we need are

$$\langle \ell k | n \rangle^2 = \langle k \ell | n \rangle^2 , \quad (A6)$$



$$\langle \ell k | n \rangle^2 = \left( \frac{2n+1}{2k+1} \right) \langle \ell n | k \rangle^2, \quad (A7)$$

$$\langle \ell k | 0 \rangle^2 = \frac{1}{2\ell+1} \delta_{\ell k}, \quad (A8)$$

$$\begin{aligned} \langle \ell 1 | n \rangle^2 &= \frac{\ell+1}{2\ell+1}, & n &= \ell + 1 \\ &= \frac{\ell}{2\ell+1}, & n &= \ell - 1 \\ &= 0. & \text{For other values of } n. \end{aligned} \quad (A9)$$

The result given in (A4) can be used to reduce more complicated products such as

$$\begin{aligned} P_k P_\ell P_n &= \sum_{n'} \langle k \ell | n' \rangle^2 P_{n'} P_n \\ &= \sum_{n', n''} \langle k \ell | n' \rangle^2 \langle n' n | n'' \rangle^2 P_{n''}, \end{aligned} \quad (A10)$$

which gives the result (A5) if both sides are integrated over  $\mu$ , and (A1) with (A8) is used.

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